# Foundations of Analog Algorithms

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**Abstract.** We propose a formalization of analog algorithms, extending the framework of abstract state machines to continuous-time models of computation.

*The machine to be described here, like almost every contrivance, apparatus, or machine in practical use, is based very largely upon what has been accomplished by others who previously labored in the same field.*
—Description of the U.S. Coast and Geodetic Survey tide-predicting machine, no. 2 (1915)

## 1 Introduction

*Abstract state machines (ASMs)* [12] constitute a most general model of sequential digital computation, one that can operate on any level of abstraction of data structures and native operations. By virtue of the Abstract State Machine Theorem of [13], any algorithm that satisfies three "Sequential Postulates" can be step-by-step emulated by an ASM. These postulates formalize the following intuitions: (I) we are talking about deterministic state-transition systems; (II) the information in states suffices to determine future transitions and may be captured by logical structures that respect isomorphisms; and (III) transitions are governed by the values of a finite and input-independent set of (variable-free) terms.

All notions of algorithms for classical discrete-time models of computation in computer science, like Turing machines, random-access memory (RAM) machines, as well as classical extensions of them, including oracle Turing machines, alternating Turing machines, and the like, [13] fall under the purview of the Sequential Postulates. This provides a basis for deriving computability theory, or even complexity theory, upon these very basic axioms about what an algorithm really is. In particular, adding a fourth axiom about initial states, yields a way to derive a proof of the Church-Turing Thesis [4,10,5].

Our goal in the current work is to adapt and extend ideas from work on ASMs to the analog case, that is to say, from notions of algorithms for digital models or systems to *analog systems*.

We do not want to deal here only with the issue of "continuous space", that is, discrete-time models or algorithms with real-valued operations, since these already fit comfortably within the standard ASM framework. See [1,2]. Indeed, algorithms for discrete-time analog models, like algorithms for the Blum-Shub-Smale model of computation [3], can also be covered by the settings of [13].

We want to deal with truly analog systems, that is to say continuous space and time systems. As surveyed in [7], several approaches have led to continuous-time models of computations. In particular, one approach inspired by continuous-time analog machines, has its roots in models of natural or artificial analog machinery. An alternate approach, one that can be refered to as inspired by continuous-time system theories, is broader in scope, and derives from research on continuous-time systems theory from a computational perspective. Hybrid systems and automata theory, for example, are two such sources.

At its beginning, continuous-time computation theory was mainly concerned with analog machines. Determining which systems can actually be considered as computational models is a very intriguing question. This relates to the philosophical discussion about what is a programmable machine. Nonetheless, there are some early examples of built analog devices that are generally accepted as programmable machines. They include Pascal's 1642 *Pascaline* [9], Hermann's 1814 *Planimeter*, Bush's landmark 1931 *Differential Analyzer* [6], as well as Bill Phillips' 1949 water-run *Financephalograph* [18]. Continuous-time computational models also include neural networks and systems that can be built using electronic analog devices. Such systems begin in some initial state and evolves over time in response to some input signal. Results are read off from the evolving state and/or from a terminal state.

Another line of development of continuous-time computation models has been motivated by hybrid systems, particularly by questions related to the hardness of their verification and control. Here models are not seen as models of necessarily analog machines, but as abstraction of systems about which one would like to establish some properties or derive verification algorithms.

Our aim is here to cover here all these models, with a uniform notion of computation and of algorithm.

We believe capturing the notion of algorithm or computation for analog systems is a first step towards a better understanding of computability theory for continuous-time systems. We refer to [7] for a survey and discussion on continuous-time computability theories.

Even this first step is a non-trivial task. Some work in this direction has been done for simple signals. See, for example, [8]. Simple (loop-free) examples are the geometric algorithms in [15]. An interesting approach to specifying some continuous-time evolutions, based on abstract state machines and using infinitesimals, is [16]. However, we believe that a general framework capturing general analog systems is wanting.

The rest of this paper is organized as follows. In the next section, we introduce dynamical transition systems, defining signals and transition systems. In Section 3, we introduce abstract dynamical systems. Then, in Section 4, we

define what an algorithmic dynamical system is. Finally, in Section 5, we define analog programs and provide some examples.

## 2 Dynamical Transition Systems

Analog systems can be thought of as "states" that evolve over "time". The systems we deal with receive inputs, called "signals", but do not otherwise interact with their environment.

### 2.1 Signals

Typically, a signal is a function from an interval of time to a "domain" value, or to a tuple of atomic domain values. For simplicity, we will presume that signals are indexed by real-valued time $\mathbb{T} = \mathbb{R}$, are defined only for a finite initial (open or closed) segment of $\mathbb{T}$, and take values in some domain $D$. Usually, the domain is more complicated than simple real numbers; it could be something like a tuple of infinitesimal signals. Every signal $u : \mathbb{T} \rightharpoonup D$ has a *length*, denoted $|u|$, such that $u(j)$ is undefined beyond $|u|$. To be more precise, the length of signals that are defined on any of the intervals $(0, \ell), [0, \ell), (0, \ell], [0, \ell]$ is $\ell$. In particular, the length of the (always undefined) *empty* signal, $\varepsilon$, is 0, as is the length of any point signal, defined only at moment 0.

The *concatenation* of signals is denoted by juxtaposition, and is defined as expected, except that concatenation of a right-closed signal with a left-closed one is only defined if they agree on the signal value at those closed ends. The empty signal $\varepsilon$ is a neutral element of the concatenation operation.

Let $\mathcal{U}$ be the set of signals for some particular domain $D$. The *prefix* relation on signals, $u \leq v$, holds if there is a $w \in \mathcal{U}$ such that $v = u\,w$. As usual, we write $u < v$ for *proper* prefixes ($u \leq v$ but $u \neq v$). It follows that $\varepsilon \leq u \leq uw$ for all signals $u, w \in \mathcal{U}$. And, $u \leq v$ implies $|u| \leq |v|$, for all $u, v$.

### 2.2 Transition Systems

**Definition 1 (Transition System).** *A* transition system $\langle \mathcal{S}, \mathcal{S}_0, \mathcal{U}, \mathcal{T} \rangle$ *consists of the following:*

- *A nonempty set (or class)* $\mathcal{S}$ *of* states *with a nonempty subset (or subclass)* $\mathcal{S}_0 \subseteq \mathcal{S}$ *of* initial *states.*
- *A set* $\mathcal{U}$ *of input signals over some domain* $D$*.*
- *A* $\mathcal{U}$*-indexed family* $\mathcal{T} = \{\tau_u\}_{u \in \mathcal{U}}$ *of* state transformations $\tau_u : \mathcal{S} \to \mathcal{S}$*.*

It will be convenient to abbreviate $\tau_u(X)$ as just $X_u$, the state of the system after receiving the signal $u$, having started in state $X$. We will also use $X_{\widetilde{u}}$ as an abbreviation for the *trajectory* $\{X_v\}_{v<u}$, describing the past evolution of the state.

For simplicity, we are assuming that the system is deterministic. Notice that the ASM framework, that is to say the classical ASM framework for digital

algorithms, initialy defined for deterministic systems, has latter been extended to nondeterministic transitions in [14,11].

**Definition 2 (Dynamical System).** *A* dynamical system $\langle \mathcal{S}, \mathcal{S}_0, \mathcal{U}, \mathcal{T} \rangle$ *is a transition system, where the transformations satisfy*

$$\tau_{uv} = \tau_v \circ \tau_u, \tag{1}$$

*for all* $u, v \in \mathcal{U}$*, and where* $\tau_\varepsilon$ *is the identity function on states.*

This implies that $X_{uv} = (X_u)_v$.

*Remark 1.* It follows from this definition that $\tau_{(uv)w} = \tau_{u(vw)}$, since composition is associative. It also follows that $\tau_a \circ \tau_a = \tau_a$, for point signal $a$, since then $aa = a$.

*Timed Transitions* *Timed transition systems* are a special case, where signals are the identity function and $D = \mathbb{T}$.

## 3 Abstract Dynamical Systems

### 3.1 Abstract States

A vocabulary $\mathcal{V}$ is a finite collection of fixed arity function symbols, some of which may be marked *relational*. A term whose outermost function name is relational is termed *Boolean*.

**Definition 3 (Abstract Transition System).** *An* abstract transition system *is a dynamical transition system whose states* $\mathcal{S}$ *are (first-order) structures over some finite vocabulary* $\mathcal{V}$*, such that the following hold:*

(*a*) *States are closed under isomorphism, so if* $X \in \mathcal{S}$ *is a state of the system, then any structure* $Y$ *isomorphic to* $X$ *is also a state in* $\mathcal{S}$*, and* $Y$ *is an initial state if* $X$ *is.*
(*b*) *Input signals are closed under isomorphism, so if* $u \in \mathcal{U}$ *is a signal of the system, then any signal* $v$ *isomorphic to* $u$ *(that is, maps to isomorphic values) is also a signal in* $\mathcal{U}$*.*
(*c*) *Transformations preserve the domain (base set); that is,* $\mathrm{Dom}\ X_u = \mathrm{Dom}\ X$ *for every state* $X \in \mathcal{S}$ *and signal* $u \in \mathcal{U}$*.*
(*d*) *Transformations respect isomorphisms, so, if* $X \cong_\zeta Y$ *is an isomorphism of states* $X, Y \in \mathcal{S}$*, and* $u \cong_\zeta v$ *is the corresponding isomorphism of input signals* $u, v \in \mathcal{U}$*, then* $X_u \cong_\zeta Y_v$*.*

In particular, system evolution is *causal* ("retrospective"): a state at any given moment is completely determined by past history and the current input signal. This is analogous to the Abstract State Postulate for discrete algorithms, as formulated by Gurevich, except that subsequent states $X_u$ depend on the whole signal $u$, not just the prior state $X$ and current input.

To keep matters simple, we are assuming (unrealistically) that all operations are total. Instead, we simply model partiality by including some *undefined* element $\perp$ in domains, as in most of the ASM literature. See however discussions in [1,2].

*Vocabularies* We will assume that the vocabularies of all states include the Boolean truth constants, the standard Boolean operations, equality, and function composition, and that these are always given their standard interpretations. We treat predicates as truth-valued functions, so states may be viewed as algebras.

There are idealized models of computation with reals, such as the BSS model [3], for which true equality of reals is available in all states. On the other hand, there are also models of computable reals, for which "numbers" are functions that approximate the idealized number to any desired degree of accuracy, and in which only partial equality is available. See [1,2] for how to extend the abstract-state-machine framework to deal faithfully with such cases.

### 3.2 Locations in States

*Locations* Since a state $X$ is a structure, it interprets function symbols in $\mathcal{V}$, assigning a value $b$ from Dom $X$ to the "location" $f(a_1, \ldots, a_k)$ in $X$ for every $k$-ary symbol $f \in \mathcal{V}$ and values $a_1, \ldots, a_k$ taken from Dom $X$. In this way, state $X$ assigns a value $[\![t]\!]_X \in$ Dom $X$ to any ground term $t$ over $\mathcal{V}$. Similarly, a state $X$ assigns the appropriate function value $[\![f]\!]_X$ to each symbol $f \in \mathcal{V}$.

*States* It is convenient to view each state as a collection of the graphs of its operations, given in the form of a set of location-value pairs, each written conventionally as $f(a_1, \ldots, a_k) \mapsto b$, for $a_1, \ldots, a_k, b \in$ Dom $X$. This allows one to apply set operations to states.

### 3.3 Updates of States

We need to capture the changes to a state that are engendered by a system. For a given abstract transition system, define its *update function* $\Delta$ as follows:

$$\Delta(X) = \lambda u.\ X_u \setminus X$$

We write $\Delta_u(X)$ for $\Delta(X)(u)$. The trajectory of a system may be recovered from its update function, as follows:

$$X_u = (X \setminus \nabla_u(X)) \cup \Delta_u(X) \tag{2}$$

where

$$\nabla_u(X) := \{\ell \mapsto [\![\ell]\!]_X : \ell \mapsto b \in \Delta_u(X) \text{ for some } b\}$$

are the location-value pairs in $X$ that are updated by $\Delta_u$.

## 4 Algorithmic Dynamic Systems

We say that states $X$ and $Y$ *agree*, with respect to a set of terms $T$, if $[\![s]\!]_X = [\![s]\!]_Y$ for all $s \in T$. This will be abbreviated $X =_T Y$. We also say that states $X$ and $Y$ are *similar*, with respect to a set of terms $T$, if or all terms $s, t \in T$, we have $[\![s]\!]_X = [\![t]\!]_X$ iff $[\![s]\!]_Y = [\![t]\!]_Y$. This will be abbreviated $X \sim_T Y$.

### 4.1 Algorithmicity

**Definition 4 (Algorithmic Transitions).** *An abstract transition system with states $\mathcal{S}$ over vocabulary $\mathcal{V}$ is* algorithmic *if there is a fixed* finite *set $T$ of* critical *terms over $\mathcal{V}$, such that $\Delta_u(X) = \Delta_u(Y)$ for any two of its states $X, Y \in \mathcal{S}$ and signal $u \in \mathcal{U}$, whenever $X$ and $Y$ agree on $T$. In symbols:*

$$X =_T Y \Rightarrow \Delta_u(X) = \Delta_u(Y) \tag{3}$$

*This implies*

$$X_{\widetilde{u}} =_T Y_{\widetilde{u}} \Rightarrow \Delta_u(X) = \Delta_u(Y) \tag{4}$$

*Furthermore, similarity should be preserved:*

$$X_{\widetilde{u}} \sim_T Y_{\widetilde{v}} \Rightarrow X_{ua} \sim_T Y_{va} \tag{5}$$

*where $a \in \mathcal{U}$ is any point signal ($|a| = 0$).*

Following the reasoning in [13, Lemma 6.2], every new value assigned by $\Delta_u(X)$ to a location in state $X$ is the value of some critical term. That is, if $\ell \mapsto b \in \Delta_u(X)$, then $b = [\![t]\!]_X$ for some critical $t \in T$.

Agreeability of states is preserved by algorithmic transitions:

**Lemma 1.** *For an algorithmic transition system with critical terms $T$, it is the case that*

$$X =_T Y \Rightarrow X_u =_T Y_u \tag{6}$$

*for any states $X, Y \in \mathcal{S}$ and input signal $u \in \mathcal{U}$.*

### 4.2 Flows and Jumps

A "jump" in a trajectory is a change in the dynamics of the system, in apposition to "flows", during which the dynamics is fixed. Formally, a jump corresponds to a change in the equivalences between critical terms, whereas, when the trajectory "flows", equivalences between critical terms is kept invariant. Accordingly, we will say that a trajectory $X_{\widetilde{u}}$ *flows* if all intermediate states $X_w$ and $X_v$ ($\epsilon < w < v < u$) are similar. It *jumps* at its end if there is no prefix $w < u$ such that all intermediate $X_v$, $w < v < u$, are similar to $X_u$. It *jumps* at its beginning if there is no prefix $w \leq u$ such that all intermediate $X_v$, $\epsilon < v < w$, are similar to $X$.

### 4.3 Analgorithms

**Definition 5 (Analog Algorithm).** *An* analog algorithm *(or "analgorithm") is an algorithmic (abstract) transition system, such that no trajectory has more than a finite number of (prefixes that end in) jumps.*

In other words, an analog algorithm is a signal-indexed deterministic state-transition system (Definitions 1 and 2), whose states are algebras that respect isomorphisms (Definition 3), whose transitions are governed by the values of a fixed finite set of terms (Definition 4), and whose trajectories do not change dynamics infinitely often (Definition 5).

### 4.4 Properties

System evolution is *causal* ("retrospective"): a state at any given moment is completely determined by past history and the current input signal.

**Theorem 1.** *For any analog algorithm, the trajectory can be recovered from the immediate past (or updates from the past). That is, $X_u$, for right-closed signal $u$, can be obtained (up to isomorphism) as a function of $X_{\widetilde{u}}$ (that is, the $X_v$, for $v < u$) plus the final input $u_*$.*

In fact, $X_u$ depends on arbitrarily small segments $X_{u(t,|u|)}$ $(t < |u|)$ of past history.

## 5 Programs

### 5.1 Definition

**Definition 6 (ASM).** *An* ASM program *$P$ over a vocabulary $\mathcal{V}$ is a finite text, taking one of the following forms:*

- *A* constraint *statement $v_1, \ldots, v_n$* **such that** *$C$, where $C$ is a Boolean condition over $\mathcal{V}$ and the $v_i$ are terms over $\mathcal{V}$ (usually subterms of $C$) whose values may change in connection with execution of this statement.*
- *A* parallel *statement $[P_1 \parallel \cdots \parallel P_n]$ ($n \geq 0$), where each of the $P_i$ is an ASM program over $\mathcal{V}$. (If $n = 0$, this is "do nothing" or "skip".)*
- *A* conditional *statement* **if** *$C$* **then** *$P$, where $C$ is a Boolean condition over $\mathcal{V}$, and $P$ is an ASM program over $\mathcal{V}$.*

We can use an assignment statement $f(s_1, \ldots, s_n) := t$ as an abbreviation for $f(s_1, \ldots, s_n)$ **such that** $f(s_1, \ldots, s_n) = t$. But bear in mind that the result is instantaneous, so that $x := 2x$ is tantamount to $x := 0$, regardless of the prior value of $x$. Similarly, $x := x + 1$ is only possible if the domain includes an "infinite" value $\infty$ for which $\infty = \infty + 1$.

### 5.2 Examples

We restrict in a first step to analog algorithms that purely flow, that is to say with no jump.

In simple continuous-time systems, the state evolves continually, governed by ordinary differential equations, say. Flow programs invoke a time parameter, which we assume is supplied by the input signal.

*Example 1 (Pendulum).* The motion of an idealized simple pendulum is governed by the second-order differential equation

$$\theta'' + \frac{g}{L}\theta = 0$$

where $\theta$ is angular displacement, $g$ is gravitational acceleration, and $L$ is the length of the pendulum rod. Let the signal $u \in \mathcal{U}$ be just real time. States report the current angle $\theta \in \mathcal{V}$. All states are endowed with the same (or isomorphic) operations for real arithmetic, including sine and square root, interpreting standard symbols. Initial states contain values for $g$, $L$, and the initial angle $\theta_0$ when the pendulum is released.

For small $\theta_0$, the flow trajectory $\tau_t(X)$ can be specified simply by

$$\theta = \theta_0 \cdot \sin\left(\sqrt{\frac{g}{L}} \cdot \imath\right)$$

where $\imath$ is the input port and nothing but $\theta$ changes from state to state. The update function is, accordingly,

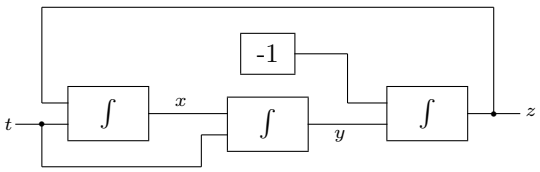
$$\Delta_t(X) = \left\{\theta \mapsto \theta_0 \cdot \sin\left(\sqrt{\frac{g}{L}} \cdot \imath\right)\right\}$$

Hence, the critical term is $\theta_0 \cdot \sin(\sqrt{g/L} \cdot \imath)$.

It can be described by program

$$[\theta \textbf{ such that } \theta = \theta_0 \cdot \sin\left(\sqrt{\frac{g}{L}} \cdot \imath\right)]$$

*Example 2 (GPAC).* One of the most famous models of analog computations is the General Purpose Analog Computer (GPAC) of Claude Shannon [17]. Here is a (non-mimimal) GPAC that generates sine and cosine: in this picture, the $\int$ signs denote some integrator, and the $-1$ denote some constant block.




If initial conditions are set up correctly, such a system will evolve according to the following initial value problem

$$\begin{cases} x' = z & x(0) = 1 \\ y' = x & y(0) = 0 \\ z' = -y & z(0) = 0\,, \end{cases}$$

It follows that $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z = -\sin(t)$.

In other words, this simple GPAC that generates sine and cosine can be modeled implicitly as a system with initial state having $x = 1; y = 0; z = 0$ and by a program

$$[x, y, z \textbf{ such that } x' = z \wedge y' = x \wedge z' = x]$$

where we presume that $x', y', z'$ denote derivatives of corresponding functions.

The proposed model can also adequately describe systems (like a bouncing ball) in which the dynamics change periodically:

*Example 3.* The physics of a bouncing ball are given by the explicit flow equations

$$v = v_0 - g \cdot t$$
$$x = v \cdot t$$

where $g$ is the gravitational constant, $v_0$ is the velocity when last hitting the table, and $t$ is the time signal—except that upon impact, each time $x = 0$, the velocity changes according to

$$v_0 = -k \cdot v$$

where $k$ is the coefficient of impact. The critical Boolean term is $x = 0$. In any finite time interval, this condition changes value only finitely many times. □

It can be described by a program like

$$[\textbf{if } x \neq 0 \textbf{ then } x, v \textbf{ such that } v = v_0 - g \cdot t, x = v \cdot t \parallel \textbf{if } x = 0 \textbf{ then } v_0 := -k \cdot v],$$

where $x$ stands for its height, $v$ its speeed. Every time the ball bounces, its speed is reduced by a factor $k$.

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